Linear Beam Dynamics and Beyond

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I INTRODUCTION

The objective of this lecture is to give a survey of the tools needed for linear lattice analysis. Although known for a long time [1] and widely available in textbooks, e.g. [2–6], linear beam dynamics presented in an entire view may be useful in the framework of this School to provide reference material for subsequent lectures and tutorial courses.

Among different formulations of the betatron motion theory, we prefer the complex Floquet function formalism [2], and the examples of linear and nonlinear dynamics problems presented below are intended to illustrate its efficiency.

II EQUATIONS OF MOTION

We consider the design orbit in a circular accelerator as a closed planar curve made up of arcs, each with a constant radius of curvature r_0. Each bend in the orbit is caused by a sector magnet, see Fig. 1a. The cylindrical coordinates r, θ, z are applied in each sector magnet (Fig. 1b), and its magnetic field $\mathbf{B} = (B_r,\, B_\theta,\, B_z)$ is two-dimensional:

$$B_\theta = 0, \quad \frac{\partial B_{r,z}}{\partial \theta} = 0.$$

We start with the relativistic equation of motion in the horizontal plane

$$\gamma m \ddot{r} = \gamma m \frac{v_\theta^2}{r} + \frac{e}{c} B_z v_\theta \tag{1}$$

with the centrifugal and Lorentz forces on the right side. Here e and m are the particle's charge and mass, $\mathbf{v} = (v_r,\, v_\theta,\, v_z)$ is its velocity, c is the speed of light, $\beta = v/c$ and $\gamma = (1 - v^2/c^2)^{-1/2}$ are the relativistic factors. On the design orbit the vertical field B_0 is related to the nominal momentum p_0,

$$p_0 c = \gamma_0 \beta_0 m c^2 = -e B_0 r_0, \tag{2}$$

where the subscript zero indicates nominal particle parameters. Consider a particle's trajectory in a neighborhood close to the design orbit and use the development around the design orbit

$$v_{z,r} \ll v_\theta \approx \beta_0 c, \quad x = r - r_0 \ll r_0\,.$$

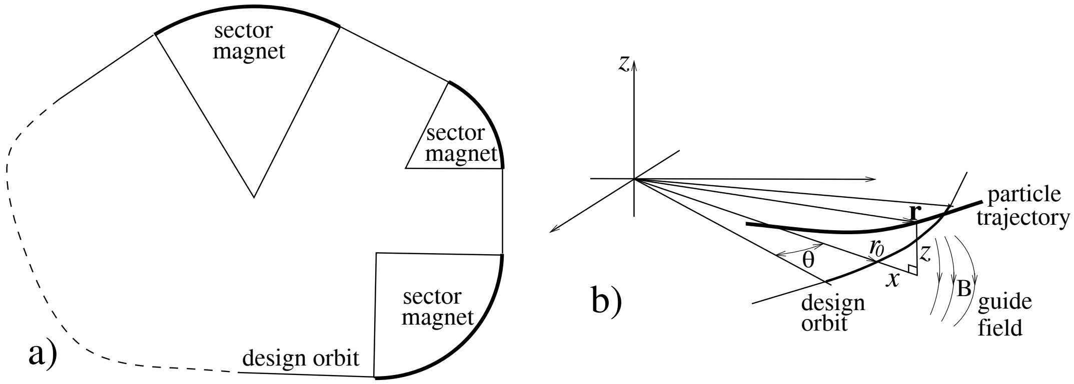

FIGURE 1. a) Design orbit formed by sector magnets; b) accelerator coordinates x, z describe particle trajectories using the design orbit as a reference.

Instead of equations of motion it is convenient to use equations of trajectories, changing the independent variable in Eq. (1) from time t to the path s along the design orbit, $ds = v_\theta dt \approx v_0 dt$,

$$r'' = \frac{1}{r} + \frac{eB_z}{\gamma\beta mc^2}, \tag{3}$$

where the prime indicates the derivative over s. Development to first order on the right side of Eq. (3), taking account of the particle's momentum offset $p = p_0 + \Delta p$, and relation (2) yields

$$\frac{1}{r_0 + x} + \frac{e\,(B_0 + \partial B_z/\partial r\, x + \ldots)}{\gamma_0\beta_0 mc^2(1 + \Delta p/p_0)} \approx -\frac{x}{r_0^2} + \frac{e}{p_0 c}\frac{\partial B_z}{\partial r}x + \frac{1}{r_0}\frac{\Delta p}{p}.$$

Thus we have obtained the linearized equation of horizontal motion,

$$x'' + K_x x = \frac{1}{r_0}\frac{\Delta p}{p}, \tag{4}$$

with the horizontal focusing function K_x in terms of the orbit curvature, and the guide field gradient calculated on the design orbit,

$$K_x = \frac{1}{r_0^2} - \frac{e}{p_0 c}\frac{\partial B_z}{\partial r}. \tag{5}$$

Similar development for the equation of vertical motion

$$\gamma m\ddot{z} = -\frac{e}{c}B_r v_\theta\,,$$

taking account of

$$B_r = 0 + \frac{\partial B_r}{\partial z}z + \ldots \approx \frac{\partial B_z}{\partial r}z, \qquad (\mathrm{rot}\mathbf{B})_\theta = \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} = 0,$$

yields the linearized equation of vertical motion

$$z'' + K_z z = 0, \tag{6}$$

where K_z is the vertical focusing function,

$$K_z = \frac{e}{p_0 c} \frac{\partial B_z}{\partial r}. \tag{7}$$

A Dispersion

Consider a closed trajectory for an off-momentum particle with $p = p_0 + \Delta p$, and call it the off-momentum orbit.

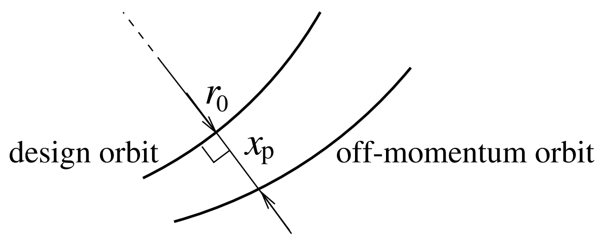

FIGURE 2. Deviation x_p of the closed orbit for off-momentum particles from the design orbit.

This orbit deviates from the design orbit by x_p, as shown in Fig. 2,

$$x_p = D_x \frac{\Delta p}{p} \tag{8}$$

where D_x is called the (horizontal) dispersion. It should to be found, after substituting Eq. (8) into Eq. (4), as a periodic particular solution of the linearized horizontal equation,

$$D_x'' + K_x D_x = \frac{1}{r_0}, \tag{9}$$

where the focusing function $K_x(s)$ and the design orbit curvature radius $r_0(s)$ are periodic functions with the period C_0 of the design orbit circumference.

Dispersion results in first-order path lengthening along the off-momentum orbit,

$$ds\left(1 + \frac{D_x}{r_0}\frac{\Delta p}{p}\right) - ds = \frac{D_x}{r_0}\frac{\Delta p}{p} ds. \tag{10}$$

Integration of Eq. (10) around the machine gives for the off-momentum orbit circumference C,

$$C - C_0 = \Delta C = \frac{\Delta p}{p} \oint \frac{D_x}{r_0} ds \equiv \alpha_p C_0 \frac{\Delta p}{p}, \tag{11}$$

where we introduced the momentum compaction factor

$$\alpha_p = \frac{1}{C_0} \oint \frac{D_x}{r_0} ds \,.$$

In modern strong-focusing machines $\alpha_p \ll 1$ (a simple estimate is $\alpha_p \approx 1/Q_x^2$, where Q_x is the horizontal betatron oscillation tune).

For the off-momentum particle revolution period T we take into account both the off-momentum orbit lengthening, Eq. (11), and the deviation of the off-momentum velocity $\beta = \beta_0 + \Delta\beta$ from the nominal one,

$$T = \frac{C}{\beta c} = \frac{C_0(1 + \Delta C/C_0)}{\beta_0(1 + \Delta\beta/\beta_0)} \approx T_0 \left(1 + \frac{\Delta C}{C} - \frac{\Delta\beta}{\beta}\right) . \tag{12}$$

Using the momentum compaction α_p and relating the particle's velocity to its momentum,

$$\frac{\Delta p}{p} = \frac{\Delta(\gamma m v)}{p} = \frac{1}{\gamma m v} \Delta \left(\frac{mv}{\sqrt{1 - v^2/c^2}}\right) = \gamma^2 \frac{\Delta v}{v} \,,$$

we can express the deviation of the revolution frequency ω_0 for the off-momentum particle from Eq. (12),

$$\frac{\Delta\omega_0}{\omega_0} = -\frac{T - T_0}{T_0} = -\left(\alpha_p - \frac{1}{\gamma^2}\right) \frac{\Delta p}{p} \equiv \eta \frac{\Delta p}{p},$$

where we defined the slippage factor η, which plays an essential role in the longitudinal dynamics of particles, considered elsewhere [4].

We deal hereafter only with the transverse motion of on-momentum particles.

III HILL'S EQUATION

Modern focusing systems of circular accelerators are composed of complicated combinations of focusing magnets. To reveal the general properties of the transverse motion, we need to study first the linearized equations with variable focusing functions $K(s)$,

$$x'' + K(s)\, x = 0, \tag{13}$$

with the single restriction that this function is apparently periodic with the machine orbit circumference C, i.e. $K(s + C) = K(s)$. This equation is called Hill's equation; our Eqs. (4) and (6) belong to this type.

A Constant Focusing

Consider first the simple special case of $K = \mathrm{const}$. For $K > 0$, we can take the cosine trajectory,

$$\mathcal{C}(s) = \cos\sqrt{K}s\,, \begin{cases} \mathcal{C}(0) = 1 \\ \mathcal{C}'(0) = 0 \end{cases}$$

and the sine trajectory,

$$\mathcal{S}(s) = \sin\sqrt{K}s\,, \begin{cases} \mathcal{S}(0) = 0 \\ \mathcal{S}'(0) = 1 \end{cases}$$

as a complete set of two linearly independent particular solutions of Hill's equation, Eq. (13), which yields simple harmonic oscillations in this special case.

For $K < 0$, we should replace the above solutions by the respective hyperbolic functions,

$$\mathcal{C}(s) = \cosh\sqrt{-K}s\,, \begin{cases} \mathcal{C}(0) = 1 \\ \mathcal{C}'(0) = 0 \end{cases}, \qquad \mathcal{S}(s) = \sinh\sqrt{-K}s\,, \begin{cases} \mathcal{S}(0) = 0 \\ \mathcal{S}'(0) = 1 \end{cases},$$

and this motion is locally unstable, the deviations from the design orbit grow exponentially.

The chosen set of solutions provides for convenient expression of the general solution to Eq. (13), with given initial displacement x_0 and initial slope x_0' of a trajectory in the matrix form,

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} \mathcal{C}(s) & \mathcal{S}(s) \\ \mathcal{C}'(s) & \mathcal{S}'(s) \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} \equiv T \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}. \tag{14}$$

Here matrix T is called the transfer matrix, and we see that transport of a trajectory specified by its initial conditions is just a linear transformation.

B Alternating-Gradient Focusing

Since Hill's equation is a second-order linear equation, its general solution in the form given by Eq. (14) holds for arbitrary $K(s) \neq \mathrm{const}$. In this case the cosine and sine trajectories are to be found first, by solving the differential equation with the appropriate initial conditions. In practice, we can recommend approximation of $K(s)$ by step functions, to whatever detail needed, as shown in Fig. 3, then using solutions with $K(s) = \mathrm{const}$ at each interval.

When these intervals are concatenated, the continuity of the solution $x(s)$ (and of the trajectory slope $x'(s)$) is preserved by multiplication of the respective transfer matrices:

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = T_3(s|s_2)\left(T_2(s_2|s_1)\left(T_1(s_1|s_0)\begin{pmatrix} x_0 \\ x_0' \end{pmatrix}\right)\right) = T_3(s)T_2T_1\begin{pmatrix} x_0 \\ x_0' \end{pmatrix}.$$

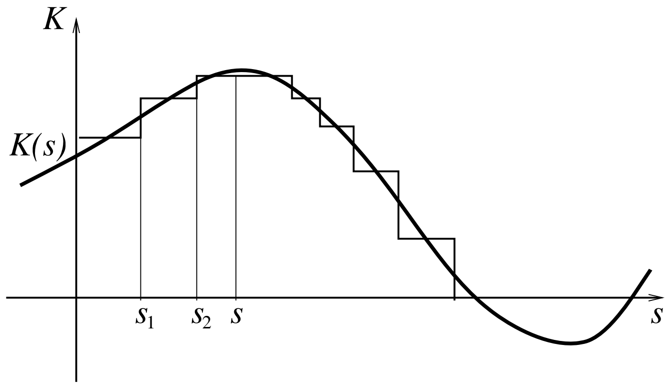

FIGURE 3. Approximation of $K(s)$ by step functions.

The resulting transfer matrix of an arbitrary focusing system is thus available,

$$T(s|s_0) = \dots T_3 T_2 T_1 = \begin{pmatrix} \mathcal{C}(s) & \mathcal{S}(s) \\ \mathcal{C}'(s) & \mathcal{S}'(s) \end{pmatrix}. \tag{15}$$

For any T, $\det T = \mathrm{const}$, being the Wronskian of Hill's equation. Our specific choice of the initial conditions provides for $\det T = 1$.

C One-Period Matrix $M(s)$ and Stability

Now we introduce the one-period transfer matrix $M(s)$,

$$M(s) = T(s + C|s), \tag{16}$$

which transports the solution forward by one period, see Fig. 4.

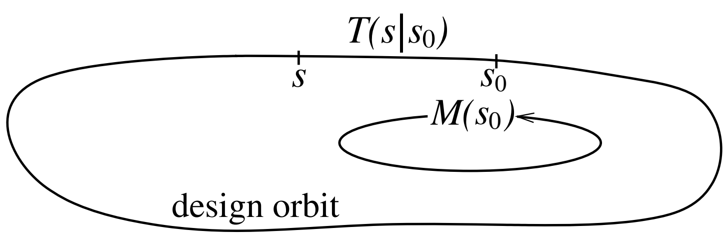

FIGURE 4. Transformation by a one-period matrix $M(s_0)$.

For stable motion, we should have limited values of x and x' when applying M repeatedly to any initial condition. Transport over N periods is given by M^N, therefore stability requires that eigenvalues λ of M must be limited, $|\lambda| \leq 1$. Otherwise, with $|\lambda| > 1$, λ^N means a possibility of unlimited growth of displacements.

Rewriting M via its matrix elements,

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix},$$

we find the eigenvalues of M from the characteristic equation

$$\det(M - \lambda I) = 0,$$

or, explicitly,

$$\left| \begin{array}{cc} m_{11} - \lambda & m_{12} \\ m_{21} & m_{22} - \lambda \end{array} \right| = \lambda^2 - (m_{11} + m_{22})\lambda + \det M = 0.$$

Using $\det M = 1$ and denoting the trace of M, $m_{11}+m_{22} = \mathrm{tr}M$, we solve this equation for λ,

$$\lambda_{1,2} = \frac{1}{2}\mathrm{tr}M \pm i\sqrt{1 - \left(\frac{1}{2}\mathrm{tr}M\right)^2} \equiv \cos\mu \pm i\sin\mu = e^{\pm i\mu}, \tag{17}$$

where $\cos\mu = \frac{1}{2}\mathrm{tr}M$. From $\det M = 1$ we immediately have $\lambda_1\lambda_2 = 1$, which means that assuming $|\lambda_1| < 1$ we would have $|\lambda_2| > 1$, i.e. growing displacements become possible with certain initial conditions. Thus, the stability condition $|\lambda| \leq 1$ is reduced to $|\lambda| = 1$ only.

In other words, $\mathrm{Im}\mu = 0$ in Eq. (17), or $|\cos\mu| \leq 1$. Finally, the stability condition can be expressed in terms of matrix M,

$$-2 \leq \mathrm{tr}M \leq 2\,. \tag{18}$$

Stable solutions of Hill's equation are called betatron oscillations, and the meaning of parameter μ is the phase advance of these oscillations over one period of the focusing structure.

D Twiss Parametrization

The one-period matrix M may be conveniently represented via the identity matrix I and the zero-trace matrix J, $\mathrm{tr}J = 0$, composed of the so-called Twiss parameters,[1]

$$M = I\cos\mu + J\sin\mu = \left(\begin{array}{cc} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{array} \right). \tag{19}$$

Among the matrix elements of J,

$$J = \left(\begin{array}{cc} \alpha & \beta \\ -\gamma & -\alpha \end{array} \right), \tag{20}$$

there are only two independent parameters, since the relation $\det M = 1$ bounds these matrix elements, $\gamma\beta - \alpha^2 = 1$, or $\det J = 1$. Hence, $J^2 = -I$, and the matrix exponent form of M follows, $M = \exp(\mu J)$.

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1) Conventionally denoted as β, γ, the Twiss parameters should not be confused with the relativistic factors.

The matrix elements of $M(s)$ are apparently periodic functions of s, $M(s+C) = M(s)$, and so are the Twiss functions $\beta(s)$, $\alpha(s)$ and $\gamma(s)$.

Provided $M(s_0)$ is known, transformation to another point s is given by the transfer matrix $T(s|s_0)$, see Fig. 4,

$$M(s) = T\, M(s_0)\, T^{-1}. \tag{21}$$

Exercise. Proof of transformation (21) is left to the reader as an exercise.

Exercise. Prove that parameter μ does not depend on s.

This transformation of M is in fact a linear transformation of its matrix elements, therefore a linear transformation of the Twiss functions from s_0 to s. Sometimes it is convenient to represent this transformation by a 3×3 matrix, called Steffen's matrix, acting on a 3-vector (β, α, γ),

$$\begin{pmatrix} \beta(s) \\ \alpha(s) \\ \gamma(s) \end{pmatrix} = \begin{pmatrix} t_{11}^2 & -2t_{11}t_{12} & t_{12}^2 \\ -t_{11}t_{21} & t_{11}t_{22} + t_{12}t_{21} & -t_{12}t_{22} \\ t_{21}^2 & -2t_{21}t_{22} & t_{22}^2 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \alpha_0 \\ \gamma_0 \end{pmatrix}, \tag{22}$$

where t_{ik} are the matrix elements of $T(s|s_0)$.

Exercise. Derive the elements of Steffen's matrix from Eq. (21).

Next we need to derive differential equations for the Twiss parameters. First define matrix D,

$$D = \begin{pmatrix} 0 & 1 \\ -K & 0 \end{pmatrix}, \tag{23}$$

which contains the focusing function K and serves for rewriting Hill's equation in matrix form,

$$X' = \frac{d}{ds} \begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -K & 0 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix} = DX\,. \tag{24}$$

The differential equation for the transfer matrix T has the same form. Indeed,

$$\frac{d}{ds}T = \begin{pmatrix} \mathcal{C}' & \mathcal{S}' \\ \mathcal{C}'' & \mathcal{S}'' \end{pmatrix} = \begin{pmatrix} \mathcal{C}' & \mathcal{S}' \\ -K\mathcal{C} & -K\mathcal{S} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -K & 0 \end{pmatrix} \begin{pmatrix} \mathcal{C} & \mathcal{S} \\ \mathcal{C}' & \mathcal{S}' \end{pmatrix},$$

or

$$T' = D\,T\,. \tag{25}$$

However, the differential equation for a one-period matrix $M(s) = T(s+C|s)$ is quite different. To derive it, we start from Eq. (21),

$$M = M(s) = T\, M_0 T^{-1},$$

rewrite it as

$$M\,T = T\,M_0\,,$$

and differentiate with respect to s,

$$M' T + M T' = T' M_0 \,.$$

Hence, using Eq. (25), we get

$$M' T + M D T = D T M_0 \,.$$

Multiplying on the right by T^{-1}, and using Eq. (21), we obtain

$$M' + MD = D T M_0 T^{-1} = DM \,,$$

or, finally,

$$M' = DM - MD \,. \tag{26}$$

Substituting here the Twiss form of M, Eq. (19), we find a set of differential equations for the Twiss functions,

$$\begin{aligned} \beta' &= -2\alpha \,, \\ \alpha' &= K\beta - \gamma \,, \\ \gamma' &= 2K\alpha \,. \end{aligned} \tag{27}$$

Elimination of α and substitution of $\gamma = (1+\alpha^2)/\beta$ yields a rather cumbersome equation for the β-function alone,

$$\frac{1}{2}\beta\beta'' - \frac{1}{4}\beta'^2 + K\beta^2 = 1 \,. \tag{28}$$

Note that these equations should be solved with periodic boundary conditions, since the Twiss functions are periodic.

Fortunately, equations for some more convenient functions, $w(s) = \sqrt{\beta(s)}$, look much better. Substituting

$$\beta = w^2, \quad \alpha = -\beta'/2 = -ww', \quad \gamma = w'^2 + \frac{1}{w^2}, \tag{29}$$

into $\alpha' = K\beta - \gamma$ in Eq. (27),

$$-(ww')' = -ww'' - w'^2 = Kw^2 - w'^2 - \frac{1}{w^2},$$

we get a nice equation for w,

$$w'' + Kw = \frac{1}{w^3}, \tag{30}$$

again with periodic boundary conditions.

Exercise. Show that the general solution of Eq. (30) in a focusing-free section ($K = 0$) has the form

$$w(s) = \sqrt{\beta_0 + \frac{(s - s_0)^2}{\beta_0}},$$

s_0 and β_0 being the constants of integration. What is their meaning?

Exercise. Find the general solution of Eq. (30) in a constant-focusing section ($K = \mathrm{const}$). Compare the result with that given by the transformation (22).

E Eigenvectors of $M(s)$

Now we find the eigenvectors $F^T = (f, f')$ of the one-period matrix $M(s)$, using its Twiss form Eq. (19) and knowing its eigenvalues $\lambda_{1,2} = e^{\pm i\mu}$. From $MF = e^{\pm i\mu}F$,

$$\begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix} \begin{pmatrix} f_\pm \\ f'_\pm \end{pmatrix} = e^{\pm i\mu} \begin{pmatrix} f_\pm \\ f'_\pm \end{pmatrix},$$

we have

$$\frac{f'_\pm}{f_\pm} = \frac{\pm i - \alpha}{\beta}. \tag{31}$$

Note that the eigenvector components are functions of s and obey Hill's equation, Eq. (13). Substituting $\alpha = -\beta'/2$ from Eq. (27) on the right side of Eq. (31), we obtain an expression of the eigenvector components via the Twiss parameters, in the form of a differential equation,

$$\frac{f'_\pm}{f_\pm} = \frac{\beta'}{2\beta} \pm \frac{i}{\beta}.$$

Integration yields a fundamental relation of the eigenvector to the β-function,

$$f_\pm(s) = f_0\sqrt{\beta(s)}\exp\left[\pm i\int^s \frac{ds'}{\beta(s')}\right], \tag{32}$$

where f_0 is the integration constant. Using freedom of normalization, we choose $f_0 = 1$ and get the complex-conjugate pair of eigenvectors, substituting Eq. (32) into Eq. (31),

$$\begin{pmatrix} \beta \\ \pm i - \alpha \end{pmatrix} \frac{e^{\pm i\psi}}{\sqrt{\beta}}, \quad \psi = \int^s \frac{ds'}{\beta(s')}. \tag{33}$$

Note that the initial phase in Eq. (33) is still left a free parameter. Using Eqs. (29), we can also write this complex-conjugate pair of normalized eigenvectors $F,\ F^*$ in terms of w and w',

$$F = \begin{pmatrix} f \\ f' \end{pmatrix} = \begin{pmatrix} w \\ w' + i/w \end{pmatrix} e^{i\psi}, \quad \psi' = \frac{1}{w^2}. \tag{34}$$

F The Floquet Theorem

Theorem. For Hill's equation

$$x'' + K(s)\,x = 0$$

where the focusing function is periodic, $K(s+C) = K(s)$, there exist normal solutions $f(s)$,

$$f'' + K(s)\,f = 0\,,$$

for which advance by one period means multiplication by a phase factor,

$$f(s+C)=e^{i\mu}f(s)\,.$$

Indeed, the above constructed eigenvectors of M, with $f(s)=w(s)e^{i\psi(s)}$, whose absolute value is a periodic function, are transformed by M when advanced by one period, and this transformation is reduced to multiplication by the eigenvalue $e^{i\mu}$ of M,

$$\left(\begin{array}{c} f \\ f' \end{array}\right)_{s+C} = M\left(\begin{array}{c} f \\ f' \end{array}\right)_{s} = e^{i\mu}\left(\begin{array}{c} f \\ f' \end{array}\right)_{s}.$$

Moreover, the phase advance is related to the amplitude function w,

$$\psi(s+C)-\psi(s)=\mu=\oint d\psi=\oint\frac{ds}{w^2}\,. \tag{35}$$

These normal solutions $f(s)=w(s)e^{i\psi(s)}$ are often called Floquet functions. We will call $F^T=(f,f')$, given by Eq. (34), the Floquet vector. Together with its complex conjugate, they form a complete basis. Any solution of Hill's equation can be decomposed in this basis,

$$\left(\begin{array}{c} x \\ x' \end{array}\right)=\frac{A}{2}\left(\begin{array}{c} f \\ f' \end{array}\right)+\frac{A^*}{2}\left(\begin{array}{c} f^* \\ f^{*\prime} \end{array}\right)=\mathrm{Re}[AF]\,. \tag{36}$$

Using the normalization condition in the Wronskian form,

$$\left|\begin{array}{cc} f & f^* \\ f' & f^{*\prime} \end{array}\right| = e^{i\psi}e^{-i\psi}\left|\begin{array}{cc} w & w \\ w'+i/w & w'-i/w \end{array}\right| = -2i\,, \tag{37}$$

we rewrite the determinant as a skew-scalar product with the help of matrix S,

$$S=\left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right). \tag{38}$$

Then

$$\left|\begin{array}{cc} f & f^* \\ f' & f^{*\prime} \end{array}\right| = \left(\begin{array}{cc} f, & f' \end{array}\right)\left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right)\left(\begin{array}{c} f^* \\ f^{*\prime} \end{array}\right) = F^T S\,F^* = -2i\,, \tag{39}$$

while $F^T S\,F=0$. These relations help to find the decomposition constant A in Eq. (36) where multiplication on the left by $F^{*T}S$ yields:

$$A=\frac{1}{i}F^{*T}S\,X=-ie^{-i\psi}\left|\begin{array}{cc} w & x \\ w'-i/w & x' \end{array}\right|. \tag{40}$$

From the fact that A is a constant determined by the initial conditions of the trajectory, follows the Courant-Snyder invariant,

$$|A|^2=(wx'-w'x)^2+\frac{x^2}{w^2}=\gamma x^2+2\alpha xx'+\beta x'^2\equiv\epsilon. \tag{41}$$

When the solution $x(s)$ is propagated in an AG focusing lattice, the quadratic form remains constant because of appropriate variation of the Twiss functions. The physical meaning of this invariant is that it is proportional to the action variable in the particle motion.

G Pseudo-Harmonic Oscillations

From Eq. (36), using Eqs. (33) and (41), we arrive at the pseudo-harmonic form of the solutions to Hill's equation,

$$x(s) = \sqrt{\epsilon\beta(s)}\cos\psi(s)\,, \tag{42}$$

$$x'(s) = -\sqrt{\frac{\epsilon}{\beta(s)}}\Big(\sin\psi(s) + \alpha(s)\cos\psi(s)\Big)\,. \tag{43}$$

Exercise. Derive Eq. (43) from Eq. (42) by differentiation, using the relation

$$\psi(s) = \int^s \frac{ds'}{\beta(s')}.$$

Exercise. Find the betatron oscillation tune Q,

$$Q = \frac{\mu}{2\pi} = \oint \frac{ds}{\beta(s)}.$$

Exercise. Show by straightforward substitution that the solution given by Eq. (42) satisfies Hill's equation, Eq. (13).

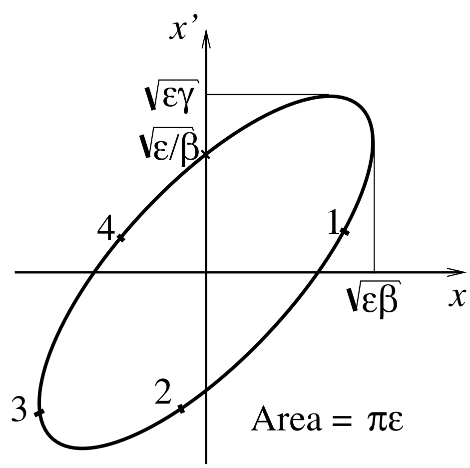

FIGURE 5. Elliptic phase-space trajectory of the betatron oscillation.

Figure 5 shows the phase space of the betatron oscillation, illustrating the meaning of the Twiss parameters. The Courant-Snyder quadratic form, Eq. (41), defines the ellipse with area $\pi\epsilon$. Being a locus of points 1, 2, ... representing one-period mapping, the ellipse is often called a phase-space trajectory of the betatron oscillation. From Fig. 5 and Eq. (42) we conclude that $w(s) = \sqrt{\beta(s)}$ is the envelope function enclosing all betatron trajectories with given $|A|$.

The pseudo-harmonic oscillation is related to the simple harmonic oscillation by a linear transformation, following from Eqs. (42) and (43),

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \sqrt{\beta} & 0 \\ -\alpha/\sqrt{\beta} & 1/\sqrt{\beta} \end{pmatrix} \begin{pmatrix} \sqrt{\epsilon}\cos\psi \\ -\sqrt{\epsilon}\sin\psi \end{pmatrix}. \tag{44}$$

and by the change of independent variable from s to ψ. The new variables are called the normalized variables.

H Perturbation of Hill's Equation

Let us add to the nominal Hill equation, Eq. (13), an additional term $g(x,s)$ on the right side,

$$x'' + K(s)\,x = g(x,s), \tag{45}$$

or, in matrix form,

$$\frac{d}{ds}\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -K & 0 \end{pmatrix}\begin{pmatrix} x \\ x' \end{pmatrix} + \begin{pmatrix} 0 \\ g \end{pmatrix}$$

or,

$$X' = DX + G\,. \tag{46}$$

The solution may be sought in the same form, Eq. (36), as before, since the nominal Floquet vectors provide for a complete basis. However, the amplitude A is no longer constant. Differentiating Eq. (40) for the amplitude A,

$$A' = \frac{1}{i}(F^{*\prime})^T S\,X + \frac{1}{i}F^{*T} S\,X'$$

using Eq. (46) for X' and the nominal Hill equation, Eq. (13), for the Floquet vectors, $F^{*\prime} = DF^*$, we have

$$A' = \frac{1}{i}F^{*T}\left(\left(D^T\,S + S\,D\right)\,X + S\,G\right).$$

Since $D^T S + SD = 0$, we finally get the equation for the variation of A caused by the perturbation

$$A' = -i\,F^{*T} S\,G\,. \tag{47}$$

Putting $A = |A|\;e^{i\phi}$, where both the absolute value and the phase of A are variable because of perturbation on the right side of Hill's equation, Eq. (45), we rewrite Eq. (47) as

$$|A|' + i\phi'\,|A| = -ie^{-i(\psi+\phi)}\,\sqrt{\beta} g(x,s)\,, \tag{48}$$

where on the right side we should put $x = |A|\,\sqrt{\beta}\cos(\psi+\phi)$, in order to complete the change of variables from x, x' to the complex amplitude $A = |A|\;e^{i\phi}$. Note that no assumptions have been made about the smallness of $g(x,s)$, and Eqs. (47) and (48) are exact equations.

IV APPLICATION OF THE FLOQUET FORMALISM

Below we present a number of problems in linear and nonlinear transverse dynamics, which are solved by means of the Floquet formalism, to illustrate its efficiency.

A Transfer Matrix in Terms of Twiss Parameters

Problem. Consider a section of a circular accelerator lattice with specified values of the Twiss parameters on its entrance and exit, i.e. β_i, α_i and β_f, α_f, respectively. Find the transfer matrix T of the section.

To solve this problem, we write the transformation of the Floquet vector performed by this optics,

$$T \begin{pmatrix} f \\ f' \end{pmatrix}_i = e^{i\phi} \begin{pmatrix} f \\ f' \end{pmatrix}_f , \tag{49}$$

where the arbitrary phase factor follows from freedom in the eigenvector normalization, see Eq. (33). The meaning of ϕ is the betatron oscillation phase advance provided by this optics; it is a free parameter.

Introducing $w_{i,f} = \sqrt{\beta_{i,f}}$ and $w'_{i,f} = -\alpha_{i,f}/w_{i,f}$, we compose matrix W from column Floquet vectors,

$$W = \begin{pmatrix} w & w \\ w' + i/w & w' - i/w \end{pmatrix},$$

then from Eq. (49) we obtain a matrix equation for unknown T,

$$T\, W_i = W_f \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}.$$

The solution is

$$T\,(f|i) = W_f \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} W_i^{-1}.$$

Calculation of the right side yields, for the matrix elements of $T(f|i)$,

$$\begin{aligned}
T(f|i) &= \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}, \\
t_{11} &= \frac{w_f}{w_i}(\cos\phi - w_i w_i' \sin\phi) = \sqrt{\frac{\beta_f}{\beta_i}}(\cos\phi + \alpha_i \sin\phi), \\
t_{12} &= w_f w_i \sin\phi = \sqrt{\beta_f \beta_i}\,\sin\phi, \\
t_{21} &= (\frac{w_f'}{w_i} - \frac{w_i'}{w_f})\cos\phi - (w_f' w_i' + \frac{1}{w_f w_i})\sin\phi \\
&= -\frac{1}{\sqrt{\beta_f \beta_i}}\left((\alpha_f - \alpha_i)\cos\phi + (1 + \alpha_f \alpha_i)\sin\phi\right),
\end{aligned}$$

$$t_{22} = \frac{w_i}{w_f}(\cos\phi + w_f w'_f \sin\phi) = \sqrt{\frac{\beta_i}{\beta_f}}(\cos\phi - \alpha_f \sin\phi),$$

where i, f indicate the entrance and exit of the optical section.

B Propagation of Mismatched β-Function

Problem. At injection, $s = 0$, initial β_i and β'_i are mismatched with the nominal values β_0 and β'_0 of the lattice. Trace $\beta(s)$ along the lattice.

Using the initial Twiss parameters, we construct the initial Floquet vector,

$$F_i = \left(\begin{array}{c} w_i \\ w'_i + i/w_i \end{array} \right),$$

and decompose F_i in the Floquet basis with the nominal w_0 and w'_0,

$$F_i = c_1 F_0 + c_2^* F_0^*\,, \quad F_0 = \left(\begin{array}{c} w_0 \\ w'_0 + i/w_0 \end{array} \right). \tag{50}$$

To find the constants, we multiply this decomposition on the left by F^{*T}. Then, using the normalization relations, see Eqs. (37) and (39),

$$F^{*T} S\, F = -(F^T S\, F^*)^T = (F^T S\, F^*)^* = 2i,$$

$$F^T S\, F = F^{*T} S\, F^* = 0,$$

we obtain for the decomposition constants,

$$c_1 = \frac{1}{2i} F_0^{*T} S\, F_i\,, \qquad c_2 = \frac{1}{2i} F_0^{*T} S\, F_i^*\,.$$

Decomposition (50) holds for any s downstream of the injection point at $s = 0$. Propagation of this initial Floquet vector is then determined by known functions of the nominal lattice, $w_0(s)$ and $\psi_0(s)$:

$$\left(\begin{array}{c} w(s) \\ w'(s) + \frac{i}{w(s)} \end{array} \right) e^{i\psi(s)} = c_1 \left(\begin{array}{c} w_0(s) \\ w'_0(s) + \frac{i}{w_0(s)} \end{array} \right) e^{i\psi_0(s)} + c_2^* \left(\begin{array}{c} w_0(s) \\ w'_0(s) - \frac{i}{w_0(s)} \end{array} \right) e^{-i\psi_0(s)},$$

where $\psi(0) = \psi_0(0) = 0$. The mismatched β-function is

$$\beta(s) = w_0^2(s) \left| c_1 e^{i\psi_0(s)} + c_2^* e^{-i\psi_0(s)} \right|^2 = \beta_0(s) \left(|c_1|^2 + |c_2|^2 + 2\mathrm{Re}\left[c_1 c_2 e^{2i\psi_0(s)} \right] \right).$$

A $\cos(2\psi_0(s) + \varphi)$ term emerges from the right side, indicating the beat of the β-function at twice the betatron tune.

C Amplitude-Dependent Tuneshift

Problem. Consider now a nonlinear perturbation $g(x,s) = q_m(s)x^m$ on the right side of Hill's equation, Eq. (45). Find the resulting correction to the betatron tune.

To solve this problem, we put $g(x,s) = q_m(s)x^m$ in Eq. (48) for the complex amplitude $A = |A|\ e^{i\phi}$,

$$|A|' + i\phi' |A| = -ie^{-i(\psi+\phi)} \sqrt{\beta} q_m(s) x^m. \tag{51}$$

The periodic function $q_m(s)$ here can be represented by its Fourier series. From Eq. (51) we see that the phase ϕ obeys the equation

$$\phi' = -\frac{q_m}{|A|} x^m \sqrt{\beta} \cos(\psi + \phi),$$

where we should substitute $x = |A| \sqrt{\beta} \cos(\psi + \phi)$,

$$\phi' = -q_m |A|^{m-1} \beta^{(m+1)/2} \cos^{m+1}(\psi + \phi)\,. \tag{52}$$

The average of the right side over fast oscillations[2] is non-vanishing for odd m. The phase ϕ on the right side may be kept constant while averaging, if the perturbation is small (which means that ϕ' is also small).

Starting with the case of perturbation of linear focusing, $m = 1$, we get from Eq. (52), after averaging,

$$\phi' = -\frac{1}{2} q_1(s) \beta(s)\,.$$

Integration of ϕ' over the orbit circumference yields contribution $\Delta\mu$ to the phase advance μ from the perturbation. Thus we can obtain the betatron tuneshift from additional focusing,

$$\Delta Q = \frac{\Delta\mu}{2\pi} = \frac{1}{2\pi} \oint \phi' ds = -\frac{1}{4\pi} \oint q_1(s)\beta(s)\, ds.$$

For the cubic nonlinearity $m = 3$, averaging of the right side of Eq. (52) results in

$$\overline{\cos^4(\psi + \phi)} = \frac{3}{8}\,. \tag{53}$$

Integration of ϕ' given by Eq. (52) with $m = 3$ yields the tuneshift

$$\Delta Q = \frac{\Delta\mu}{2\pi} = \frac{1}{2\pi} \oint \phi' ds = -\frac{3\,|A|^2}{16\pi} \oint q_3(s)\beta^2(s) ds\,,$$

where use has been made of Eq. (53). The found amplitude-dependent tuneshift is due to nonlinearity of the betatron motion, and the Twiss β-function squared is the weight function of the cubic perturbation. For mth-order nonlinearity the weight function will be $\beta^{(m+1)/2}$.

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2) The averaging is performed with the assumption that the betatron tune stays well apart from (higher-order) resonances.

D Dynamic Aperture Limitation by a Single Nonlinear Kick

Consider a nonlinear one-period map formed by a linear optics section, with a betatron phase advance of $2\pi Q$, followed by a thin nonlinear element which can be treated in the kick approximation. A fixed point of this map (other than $x = 0$, $x' = 0$) is the location in the phase space of the system, where the onset of stochasticity occurs. Thus the position of the fixed point(s) may provide a rough estimate of the available dynamic aperture, i.e. the phase space area around the origin where the regularity of motion is preserved.

Problem. Taking an example with a single sextupole kick $k_2 x^2$, as shown in Fig. 6, find the position of the period-one fixed point. In other words, the fixed point gives initial conditions for a special trajectory, i.e. the periodic one, to be found here.

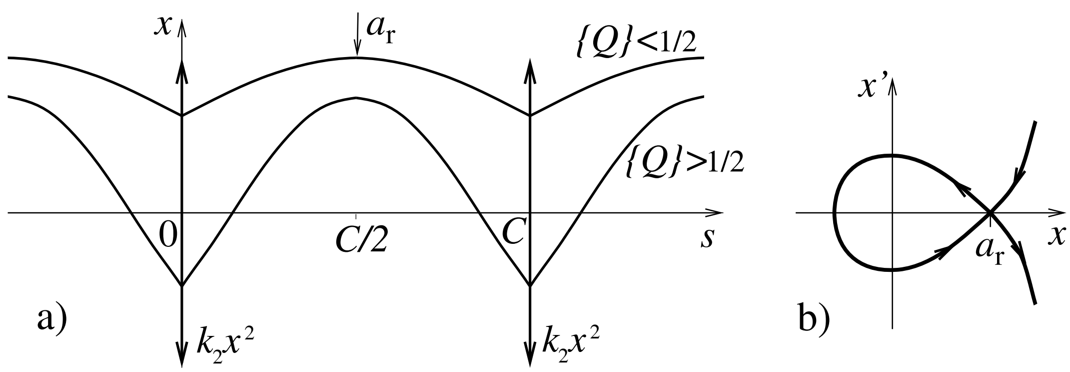

FIGURE 6. a) Periodic trajectories corresponding to the period-one fixed points of the nonlinear transformation; b) phase-space trajectory crossing the period-one fixed point a_r.

It is convenient to work in the normalized betatron variables, Eq. (44). We write the equation of the periodic trajectory on the right side of the kick, for $s > 0$,

$$x = a\cos(\psi - \pi Q)\,,$$
$$x' = -a\sin(\psi - \pi Q)\,,$$

and on the left side, for $s < 0$,

$$x = a\cos(\psi + \pi Q)\,,$$
$$x' = -a\sin(\psi + \pi Q)\,.$$

Continuity of x at $s = 0$ is provided by the form of these expressions. The slope x' is changed by the kick,

$$x'|_{+0} - x'|_{-0} = 2a_r \sin\pi Q = k_2 x_r^2 = k_2 a_r^2 \cos^2\pi Q\,,$$

and this equation determines the position a_r of the fixed point,

$$a_r = \frac{2\sin\pi Q}{k_2\cos^2\pi Q}\,.$$

The solution is markedly tune-dependent, and the neighborhood of the integer resonance should be avoided. Turning from the normalized variables back to normal ones, we see from Eq. (44) that the strength k_2 of the sextupole kick scales as $\beta^{3/2}$, where β is the value of the β-function at the kick location. Therefore, the dynamic aperture scales as $\beta^{-3/2}$.

V CONCLUSION

A versatile formalism is available (in different forms) to fully support linear lattice analysis and to simplify the formulation of nonlinear dynamics problems.

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