Space Charge

N. A. Vinokurov

Budker Institute of Nuclear Physics
Novosibirsk, Russia

Abstract. The space-charge effects in a straight beamline are considered.

INTRODUCTION

In these short notes a few simple models are described. They may be useful for obtaining rough estimates of space-charge effects, and for testing the numerical results provided by computer codes.

SPACE-CHARGE FORCES

Coasting Cylindrical Beam

1. Consider first a round coasting (i. e. unbunched, so the beam parameters do not depend on time) cylindrical (i.e. the beam parameters do not depend on the coordinate along the beam axis z) beam having charge density $\rho(r)$, where $r = \sqrt{x^2 + y^2}$ is the distance from the beam axis z. Suppose that all the particles are moving along the z axis with the same velocity v. Then the electric and magnetic fields have only one component in the cylindrical system of coordinates (r, α, z), and we can use Gauss's and Stokes's theorems to find the fields, respectively:

$$E_r = \frac{4\pi}{r}\int_0^r \rho(r')r'dr', \tag{1}$$

$$B_\alpha = \frac{v}{c}\frac{4\pi}{r}\int_0^r \rho(r')r'dr', \tag{2}$$

where c is the velocity of light. According to the derivation method, Eqs. (1) and (2) are also valid in the presence of a coaxial round cylindrical vacuum chamber. The Lorentz force has only a radial component[1]:

$$F_r = eE_r - e\beta B_z = \frac{1}{\gamma^2}eE_r, \tag{3}$$

[1] For the vacuum chamber with finite conductivity there is also the z component of electric field.

where $\beta = v/c$, and $\gamma = (1 - \beta^2)^{-1/2}$ is the relativistic factor (the ratio of the particle energy to its rest-frame energy mc^2). According to Eq. (3), the magnetic part of the Lorentz force is subtracted from the electric one, and therefore the net value of the Lorentz force decreases significantly with particle energy growth.

When a particle of a beam passes a length l, the radial component of the momentum grows as

$$\Delta p_r = F_r \frac{l}{v} = \frac{el}{v\gamma^2}\frac{4\pi}{r}\int_0^r \rho(r')r'dr'. \tag{4}$$

Since Δp_r is invariant with respect to the boost along the z axis, it is interesting to calculate it in the beam rest frame. Here only the electric field induced by the charge density ρ/γ,

$$E_r' = \frac{4\pi}{r}\int_0^r \frac{\rho(r')}{\gamma} r'dr', \tag{5}$$

exists. Because of Lorentz contraction, the force acts during a time interval $l/\gamma v$, and we again get the result of Eq. (4).

For the simplest case of homogeneous charge distribution with beam radius a,

$$\rho = \begin{cases} \dfrac{I}{v\pi a^2} & r \le a \\ 0 & r > a \end{cases}, \tag{6}$$

where I is the beam current, we have

$$E_r = \begin{cases} \dfrac{2I}{va^2} r & r \le a \\ \dfrac{2I}{vr} & r > a \end{cases}. \tag{7}$$

If the beam is propagating inside a conducting round pipe of radius b, we can choose the potential to be zero on the inner pipe surface. Then according to Eq. (1) the potential inside the beam pipe is

$$\varphi(r) = \int_r^b E_r(r')\,dr' = 4\pi \int_r^b\int_0^{r'} \rho(r'')r''dr''\frac{dr'}{r'}. \tag{8}$$

For the homogeneous charge distribution given by Eq. (6), we get

$$\varphi(r) = \begin{cases} \dfrac{2I}{v}\ln\dfrac{b}{a} + \dfrac{I}{v}\left(1 - \dfrac{r^2}{a^2}\right) & r \le a \\ \dfrac{2I}{v}\ln\dfrac{b}{r} & r > a \end{cases}. \tag{9}$$

2. For an elliptical beam cross-section,

$$\rho(x,y) = \frac{I}{v\pi ab}\vartheta\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right), \tag{10}$$

where

$$\vartheta(t)=\begin{cases}0 & t<0\\ 1 & t\geq 0\end{cases} \tag{11}$$

is the step function, the electric field inside the beam ($x^2/a^2 + y^2/b^2 \leq 1$) is also a linear function of the coordinates [1]:

$$\begin{aligned} E_x(x,y) &= \frac{4I}{v}\frac{x}{a(a+b)},\\ E_y(x,y) &= \frac{4I}{v}\frac{y}{b(a+b)}. \end{aligned} \tag{12}$$

Using Eq. (10) in the rest frame, and coming back to the laboratory frame, we can easily get the magnetic field $B_x = -\beta E_y$, $B_y = \beta E_x$ and the Lorentz force

$$\begin{aligned} F_x(x,y) &= \frac{4eI}{v\gamma^2}\frac{x}{a(a+b)},\\ F_y(x,y) &= \frac{4eI}{v\gamma^2}\frac{y}{b(a+b)}. \end{aligned} \tag{13}$$

Longitudinal Electric Field

1. Let the beam pipe have different radii b_- at $z < -L_t$ and b_+ at $z > L_t$ ($2L_t$ is the length of the transition region). Then there is a longitudinal electric field $E_z(r,z)$ in the transition region, and

$$U = -\int_{-\infty}^{\infty} E_z(r,z)dz = \varphi_+(r) - \varphi_-(r) = \frac{2I}{v}\ln\frac{b_+}{b_-}. \tag{14}$$

Note that the voltage U does not depend on r. For $b_- < b_+$ particles are decelerated, and the beam "loses" corresponding power UI. This power is deposited to the energy of the fields. Moving from left to right, the beam induces the fields in additional space, contained between the cylinders $r = b_-$ and $r = b_+$. The additional power passing through the right part of the beam pipe is

$$P = \int_{b_-}^{b_+} \frac{cE_r B_\alpha}{4\pi} 2\pi r dr = \frac{2I^2}{v}\ln\frac{b_+}{b_-}, \tag{15}$$

which is just IU. To prove energy conservation in the general case, we need to take into account the variation of the beam velocity. Then

$$U(r) = \varphi_+(r) - \varphi_-(r) = \frac{2I}{v_+}\ln\frac{b_+}{a} - \frac{2I}{v_-}\ln\frac{b_-}{a} + I\left(1-\frac{r^2}{a^2}\right)\left(\frac{1}{v_+}-\frac{1}{v_-}\right) \tag{16}$$

Averaging $U(r)$ over the beam cross-section ($\langle r^2/a^2\rangle = ½$), we get the power

$$IU = \frac{2I^2}{v_+}\left(\ln\frac{b_+}{a}+\frac{1}{4}\right)-\frac{2I^2}{v_-}\left(\ln\frac{b_-}{a}+\frac{1}{4}\right). \tag{17}$$

It can also be derived from the total energy conservation:

$$\gamma_+\frac{mc^2}{e}I+\frac{2I^2}{v_+}\left(\ln\frac{b_+}{a}+\frac{1}{4}\right)=\gamma_-\frac{mc^2}{e}I+\frac{2I^2}{v_-}\left(\ln\frac{b_-}{a}+\frac{1}{4}\right). \tag{18}$$

2. Now let the beam current depend on time as $I(t-z/v)$. Since

$$t-\frac{z}{v}=\gamma\left(t'+v\frac{z'}{c^2}\right)-\frac{\gamma}{v}\left(z'+vt'\right)=-\frac{z'}{\gamma v}, \tag{19}$$

for the ideal-conducting round beam pipe the field in the rest frame is purely electrostatic. If

$$\left|\frac{d}{dz'}\ln I\left(-\frac{z'}{\gamma v}\right)\right|=\frac{1}{\gamma v}\left|\frac{d}{dt}\ln I\left(t-\frac{z}{v}\right)\right|<<\frac{1}{b}, \tag{20}$$

i. e. the bunch in the rest frame is much longer than the pipe radius, Eqs. (8) and (9) are valid approximately. Then the corresponding longitudinal electric field in the beam ($r \le a$) is

$$E_z=-\frac{\partial}{\partial z'}\varphi'(r,z')=-\left(2\ln\frac{b}{a}+1-\frac{r^2}{a^2}\right)\frac{d}{dz'}\frac{I}{\gamma v}=\left(2\ln\frac{b}{a}+1-\frac{r^2}{a^2}\right)\frac{1}{\gamma^2v^2}\frac{\partial}{\partial t}I\left(t-\frac{z}{v}\right). \tag{21}$$

Note that E_z is Lorentz invariant, and therefore Eq. (21) is valid in the laboratory frame also. For a thin beam ($a << b$), the dependence of E_z on the transverse coordinate r inside the beam is rather weak, and we can replace r^2/a^2 by its average value ½. Another simplification of Eq. (21) can be obtained for the parabolic longitudinal distribution

$$I(t)=\begin{cases}\frac{3}{4}\frac{Q}{\tau}\left(1-\frac{t^2}{\tau^2}\right) & |t|\le\tau \\ 0 & |t|>\tau\end{cases}, \tag{22}$$

where Q and τ are the charge and the duration of the bunch. Then E_z is a linear function of $t-z/v$ inside the bunch ($|t-z/v|\le\tau$),

$$E_z=-\left(3\ln\frac{b}{a}+\frac{3}{4}\right)\frac{Q}{\gamma^2v^2\tau^3}\left(t-\frac{z}{v}\right). \tag{23}$$

3. Let the beam pipe have a small azimuthal groove with depth h and width w, i. e. the pipe surface equation is

$$r(z)=\begin{cases}b+h & |z|\le\frac{w}{2} \\ b & |z|>\frac{w}{2}\end{cases}. \tag{24}$$

For a long bunch ($\frac{w}{v}\left|\frac{d}{dt}\ln I\left(t-\frac{z}{v}\right)\right| << 1$), the magnetic field inside the groove can be approximated by Eq. (2). Therefore the magnetic flux in the groove is

$$\Phi = \frac{2Iw}{c}\ln\frac{b+h}{b} \approx \frac{2Iwh}{cb}. \tag{25}$$

Another way to obtain this flux is to calculate the magnetic field in the groove from the surface density of the image current, $i = -I/(2\pi b)$: $B_\alpha = -4\pi i/c$, $\Phi = B_\alpha wh$. According to Eq. (25), the groove inductance is $L = 2wh/b$.

As the flux is time-dependent, the corresponding voltage between the groove banks, U, is

$$-\int_{-\infty}^{\infty} E_z dz = U = \frac{1}{c}\frac{d\Phi}{dt} \approx 2\frac{wh}{c^2 b}\frac{dI(t)}{dt}. \tag{26}$$

The longitudinal impedance is defined as $Z(\omega) = U(\omega)/I(\omega)$. In our case

$$Z = -\frac{1}{c^2} i\omega L = -2i\frac{wh\omega}{c^2 b}. \tag{27}$$

Comparing Eqs. (26) and (21) we can say that the Coulomb longitudinal forces are equivalent to the effective (negative) inductance per unit length:

$$\left(\frac{L}{l}\right)_C = -\frac{2}{\gamma^2\beta^2}\left(\ln\frac{b}{a}+\frac{1}{4}\right). \tag{28}$$

TRANSVERSE MOTION

Laminar Beam

As it was shown earlier, the components of the Lorentz force inside the uniformly charged beam are linear functions of the transverse coordinates (see Eq. (13)). Let's consider a round beam. The equation of transverse motion of a particle inside such a beam has the form[2]:

$$\gamma m\frac{d^2 r_1}{dt^2} = \frac{2eI}{\gamma^2 v a^2} r_1. \tag{29}$$

If the initial particle velocities depend linearly on the initial coordinates $\frac{dr_1}{dt}(0) = Ar_1(0)$, then the linear dependence (with a time-dependent coefficient A) will be conserved during the motion, because of the linearity of Eq. (29). This means that particle trajectories do not cross. Such a beam is frequently referred to as a laminar

[2] Here and below we consider the paraxial approximation $|dr_1/dt| << v$.

beam. In the phase space (r, r'), all particles lie in the interval bounded by the points (0, 0) and (a, da/dt). Equation (29) is also valid for the boundary particles $r_1 = a$. Then, changing the independent variable $t = z/v$, we obtain

$$\frac{d^2a}{dz^2} = \frac{2I}{(\beta\gamma)^3 I_0}\frac{1}{a}, \tag{30}$$

where $I_0 = mc^3/e \approx 17$ kA (for electrons) is the characteristic current. The first integral of Eq. (30) is

$$\frac{a'^2}{2} + \frac{2I}{(\beta\gamma)^3 I_0}\ln a = \frac{2I}{(\beta\gamma)^3 I_0}\ln a_0, \tag{31}$$

where $a' = da/dz$, and a_0 is the minimum beam radius (where $a' = 0$). Using Eq. (31) we can get the expression for the angular divergence,

$$a' = \pm 2\sqrt{\frac{I}{(\beta\gamma)^3 I_0}\ln\frac{a}{a_0}}\ . \tag{32}$$

Non-relativistic beams are frequently characterized by the perveance $P = IU^{-3/2} \propto I/\beta^3$ (eU is the particle kinetic energy), then the dimensionless constant in Eq. (30) can be expressed as $P\sqrt{m/(2e)}$.

Linear external focusing can be easily taken into account by adding the corresponding force to Eq. (29). Then Eq. (30) will be generalized as follows,

$$\frac{d^2a}{dz^2} + K(z)a - \frac{2I}{(\beta\gamma)^3 I_0}\frac{1}{a} = 0, \tag{33}$$

where $K(z)$ is the focusing rigidity. For example, for a thin focusing lens installed at $z = 0$, $K(z) = \delta(z)/F$, F is the focal length, and $\delta(z)$ is Dirac's delta-function.

To generalize further our consideration of transverse motion, taking account of the space-charge force, we will discuss a beam with finite emittances.

Kapchinsky-Vladimirsky Equation

1. A particle beam can be described in more detail by using the distribution function $f(x, x', y, y')$ in the four-dimensional phase space of the transverse coordinates and angles. For a uniform elliptical beam (Eq. (10)) we can choose a distribution function of the form

$$f = \frac{1}{\pi^2\varepsilon_x\varepsilon_y}\delta\left(\frac{\beta_x x'^2 + 2\alpha_x x x' + \gamma_x x^2}{\varepsilon_x} + \frac{\beta_y y'^2 + 2\alpha_y y y' + \gamma_y y^2}{\varepsilon_y} - 1\right), \tag{34}$$

where α_x, β_x, γ_x, α_y, β_y, γ_y are z-dependent Twiss parameters, and ε_x and ε_y are constants called emittances. Distribution in Eq. (34) is referred to as the Kapchinsky-Vladimirsky (KV) distribution and corresponds to uniform particle distribution over the surface of a four-dimensional ellipsoid. Integration of the distribution function over the angles x' and y' leads to the charge density, Eq. (10), with $a^2 = \varepsilon_x\beta_x$ and $b^2 = \varepsilon_y\beta_y$. We can write the particle trajectory equations using the Lorentz force, Eq. (13),

$$x'' = -K_x(z)x + \frac{4I}{(\beta\gamma)^3 I_0}\frac{x}{a(a+b)},$$
$$y'' = -K_y(z)y + \frac{4I}{(\beta\gamma)^3 I_0}\frac{y}{b(a+b)}, \tag{35}$$

where K_x and K_y are the rigidities of the external focusing. Equation (35) is the set of two Hill's equations with total rigidities

$$K_x^{tot} = K_x(z) - \frac{4I}{(\beta\gamma)^3 I_0}\frac{1}{a(a+b)},$$
$$K_y^{tot} = K_y(z) - \frac{4I}{(\beta\gamma)^3 I_0}\frac{1}{b(a+b)}. \tag{36}$$

Then the envelope equations with these rigidities are

$$a'' + K_x(z)a - \frac{4I}{(\beta\gamma)^3 I_0}\frac{1}{a+b} - \frac{\varepsilon_x^2}{a^3} = 0,$$
$$b'' + K_y(z)b - \frac{4I}{(\beta\gamma)^3 I_0}\frac{1}{a+b} - \frac{\varepsilon_y^2}{b^3} = 0. \tag{37}$$

Equations (37) are referred to as the Kapchinsky-Vladimirsky equations. For the special distribution they reduced the problem of the evolution of the distribution function to the problem of the evolution of two transverse beam sizes a and b. It is worth noting that these are exact equations.

2. Frequently people use the so-called root-mean-square emittance

$$\varepsilon_x^{rms} = \sqrt{\langle x^2\rangle\langle x'^2\rangle - \langle x\,x'\rangle^2}\,. \tag{38}$$

To understand the origin of this combination we can write down the "symplectic distance" between two points (particles) on the phase plane

$$S = (x_1\ x_1')\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x_2 \\ x_2' \end{pmatrix} \tag{39}$$

which is the area of the parallelogram constructed on vectors $(x_1\ x'_1)$ and $(x_2\ x'_2)$. As $\langle\langle S\rangle_1\rangle_2 = 0$, we can calculate

$$\langle\langle S^2\rangle_1\rangle_2 = \langle\langle (x_1x_2')^2\rangle_1\rangle_2 - 2\langle\langle x_1x_2'x_2x_1'\rangle_1\rangle_2 + \langle\langle (x_2x_1')^2\rangle_1\rangle_2$$
$$= 2\langle x^2\rangle\langle x'^2\rangle - 2\langle x\,x'\rangle^2. \tag{40}$$

For the Kapchinsky-Vladimirsky distribution $\varepsilon_x^{rms} = \varepsilon_x/4$.

Comparison of the two nonlinear terms in Eq. (37) for a round beam, $a = b$, $\varepsilon_x = \varepsilon_y = \varepsilon$, shows that the laminar beam approximation, Eq. (33), is valid when

$$\frac{\varepsilon}{\beta_x} << \frac{2I}{(\beta\gamma)^3 I_0} \ . \tag{41}$$

The left side of Eq. (41) is the square of the local angular spread, and the right side is the dimensionless perveance, which is the square of the "characteristic angular divergence" (see Eq. (32)). In the opposite case, we can neglect the space-charge force term in Eq. (37) and return to the single-particle approximation. It is worth noting that the condition (41) depends on the beta-function. Therefore, it may be different in different places along the beamline. In particular, the places where the beta-function (and the beam size) are maximal, are the most sensitive to the influence of the space-charge forces.

Emittance Degradation

For a coasting beam and Kapchinsky-Vladimirsky transverse distribution, the space-charge force is linear and does not change the emittances. But for a bunched beam the current I depends on time. Correspondingly, at different parts of the bunch the corrections to the focusing rigidities (see Eq. (36)) are different. Therefore the Twiss parameters of the beam also become different and, averaged over the whole bunch, the transverse emittance (the so-called projection emittance) increases.

Consider a beam passing through free space of length L. According to Eq. (36), the horizontal optical strength is

$$\frac{1}{F_x} = K_x^{tot} L = -\frac{4I(s)}{(\beta\gamma)^3 I_0} \frac{L}{a(a+b)} \ , \tag{42}$$

where s is the coordinate along the bunch. The Twiss matrix transformation

$$\begin{aligned} \mathbf{J} = \mathbf{T}\mathbf{J}_0\mathbf{T}^{-1} &= \begin{pmatrix} 1 & 0 \\ -\dfrac{1}{F_x} & 1 \end{pmatrix} \begin{pmatrix} \alpha_{x0} & \beta_{x0} \\ -\gamma_{x0} & -\alpha_{x0} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \dfrac{1}{F_x} & 1 \end{pmatrix} \\ &= \begin{pmatrix} \alpha_{x0} + \dfrac{\beta_{x0}}{F_x} & \beta_{x0} \\ -\gamma_{x0} - \dfrac{2\alpha_{x0}}{F_x} - \dfrac{\beta_{x0}}{F_x^2} & -\alpha_{x0} - \dfrac{\beta_{x0}}{F_x} \end{pmatrix}, \end{aligned} \tag{43}$$

and averaging over both the transverse distribution function and the bunch length gives the rms projection emittance,

$$
\begin{aligned}
\varepsilon_x^2 &= \langle x^2\rangle\langle x'^2\rangle - \langle x x'\rangle^2 = \langle\varepsilon_{x0}\beta_x\rangle\langle\varepsilon_{x0}\gamma_x\rangle - \langle\varepsilon_{x0}\alpha_x\rangle^2 \\
&= \varepsilon_{x0}^2\left(\langle\beta_x\rangle\langle\gamma_x\rangle - \langle\alpha_x\rangle^2\right) = \varepsilon_{x0}^2\left(1 + \beta_{x0}^2\left\langle\frac{1}{F_x^2}\right\rangle - \beta_{x0}^2\left\langle\frac{1}{F_x}\right\rangle^2\right),
\end{aligned}
\tag{44}
$$

and the emittance increase,

$$
\Delta\left(\varepsilon_x^2\right) = \varepsilon_x^2 - \varepsilon_{x0}^2 = \left(\varepsilon_{x0}\beta_{x0}\right)^2\left(\left\langle\frac{1}{F_x^2}\right\rangle - \left\langle\frac{1}{F_x}\right\rangle^2\right). \tag{45}
$$

For a Gaussian bunch $I = I_{\max}\exp\left(-\frac{s^2}{2\sigma^2}\right)$, $\left\langle\frac{1}{F_x^2}\right\rangle - \left\langle\frac{1}{F_x}\right\rangle^2 = \left(\frac{1}{F_x}\right)_{\max}^2\left(\frac{1}{\sqrt{3}} - \frac{1}{2}\right)$.

Assuming for simplicity a round beam $a^2 = b^2 = 4\varepsilon_{x0}\beta_{x0}$ and a small initial emittance $\varepsilon_{x0} << \varepsilon_x$, we can easily find from Eqs. (42) and (45):

$$
\Delta\varepsilon_x \approx \frac{0.14 I_{\max} L}{(\beta\gamma)^3 I_0}. \tag{46}
$$

The other reason for emittance degradation is the nonlinearity of the Lorentz force for real transverse distributions (which differ from the Kapchinsky-Vladimirsky distribution). The contribution from nonlinearity increases the numerical coefficient in Eq. (46) from 0.14 to approximately 0.2. The emittance degradation described above is not a truly irreversible process, therefore it can be partly compensated by proper downstream optics.

LONGITUDINAL MOTION

Debunching of a Long Bunch

The equation of longitudinal motion,

$$
\frac{d}{dt}\left(\gamma m\frac{dz}{dt}\right) = eE_z, \tag{47}
$$

can be rewritten approximately as

$$
\gamma^3 m\frac{d^2 s}{dt^2} = eE_z, \tag{48}
$$

where $s = z - v_0 t$, v_0 is the average velocity and $\gamma = \left(1 - v_0^2/c^2\right)^{-1/2}$. For parabolic charge distribution, Eq. (22), the longitudinal field E_z depends on s linearly (see Eq. (23)), and we can repeat the study done above for the transverse motion of a laminar beam. Substituting Eq. (23) into Eq. (48) gives

$$\frac{d^2 s}{dt^2} = 3\left(\ln\frac{b}{a} + \frac{1}{4}\right)\frac{eQ}{\gamma^5 m l^3} s\,, \tag{49}$$

where $l = v_0\tau$. An equation for the bunch length $l = s_{max}$ follows from Eq. (49):

$$\frac{d^2 l}{dt^2} = 3\left(\ln\frac{b}{a} + \frac{1}{4}\right)\frac{eQ}{\gamma^5 m}\frac{1}{l^2}\,. \tag{50}$$

The most significant difference between Eq. (50) and Eq. (30) is the inverse square dependence of the right side. The first integral of Eq. (50) is

$$\frac{1}{2}\left(\frac{dl}{dt}\right)^2 + 3\left(\ln\frac{b}{a} + \frac{1}{4}\right)\frac{eQ}{\gamma^5 m}\frac{1}{l} = 3\left(\ln\frac{b}{a} + \frac{1}{4}\right)\frac{eQ}{\gamma^5 m}\frac{1}{l_0} = \frac{1}{2}\left(\frac{dl}{dt}\right)_\infty^2, \tag{51}$$

where l_0 is the minimum bunch length and $(dl/dt)_\infty$ is the maximum velocity deviation of the bunch end. The corresponding maximum energy deviation ($\delta\gamma = \gamma^3\beta\frac{1}{c}\frac{dl}{dt}$) is

$$(\delta\gamma)_{\max} = \gamma^3\beta\frac{1}{c}\sqrt{6\left(\ln\frac{b}{a} + \frac{1}{4}\right)\frac{eQ}{\gamma^5 m}\frac{1}{l_0}} = \sqrt{8\left(\ln\frac{b}{a} + \frac{1}{4}\right)\beta\gamma\frac{I_{\max}}{I_0}}\,. \tag{52}$$

Short-Wavelength Density Modulation

Now we'll consider the opposite limiting case of short-wavelength longitudinal density modulation. It is important for some electronic devices (klystrons, traveling-wave tubes, free electron lasers, etc.).

In the beam rest frame, space charge induces only an electric field. Therefore, the equations in hydrodynamic approximation have the form

$$\begin{aligned} &\frac{\partial\rho}{\partial t} + div(\rho\mathbf{v}) = 0\,, \\ &m\frac{d\mathbf{v}}{dt} = e\mathbf{E}\,, \\ &div\mathbf{E} = 4\pi\rho\,. \end{aligned} \tag{53}$$

Linearization

$$\begin{aligned} &\rho = \rho_0 + \rho_1\,, \\ &\frac{d\mathbf{v}}{dt} = \frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} \approx \frac{\partial\mathbf{v}}{\partial t}\,, \end{aligned} \tag{54}$$

leads to

$$\begin{aligned} &\frac{\partial\rho_1}{\partial t} + \rho_0 div\mathbf{v} = 0\,, \\ &m\frac{\partial\mathbf{v}}{\partial t} = e\mathbf{E}\,, \\ &div\mathbf{E} = 4\pi\rho_1\,, \end{aligned} \tag{55}$$

and hence to the equation for plasma oscillation,

$$\frac{\partial^2 \rho_1}{\partial t^2} + \omega_p^2 \rho_1 = 0 , \tag{56}$$

where $\omega_p = \sqrt{4\pi\rho_0 e/m}$ is the rest-frame plasma frequency. The general solution of Eq. (56) is

$$\rho_1 = A(\mathbf{r}) \cos(\omega_p t) + B(\mathbf{r}) \sin(\omega_p t) \ . \tag{57}$$

Returning to the laboratory frame, we can find the space period of the plasma oscillations,

$$L_p = \frac{2\pi\gamma\beta c}{\omega_p} = \sqrt{\pi(\beta\gamma)^3 I_0/j} \ , \tag{58}$$

where j is the current density. For uniform charge distribution, Eq. (6),

$$L_p = \pi a\sqrt{(\beta\gamma)^3 I_0/I} \ . \tag{59}$$

An interesting application of the above theory is suppression of the shot noise in the current of an electron beam from a cathode. Electrons leave the cathode at uncorrelated moments of time with approximately zero velocity spread at the output of the electron gun. After they pass length $L_p/4$, the initial density fluctuations disappear, and by accelerating the beam at this point we can "freeze" the charge distribution (L_p increases significantly after acceleration). This is how a low-noise traveling-wave tube works.

CURRENT LIMITATIONS

Energy Spread and Maximum Current

Frequently the source of an electron beam is a cathode at potential U. Then the electron total energy is eU and the kinetic energy T is (see Eq. (9))

$$T = e(U - \varphi) = e\left(U - \frac{2I}{v}\ln\frac{b}{a} + \frac{I}{v}\left(1 - \frac{r^2}{a^2}\right)\right). \tag{60}$$

The kinetic energy spread (i. e., the energy difference for particles at $r = 0$ and $r = a$) is

$$\Delta\gamma = \frac{\Delta T}{mc^2} = e(U - \varphi) = \frac{I}{\beta I_0} . \tag{61}$$

Since the kinetic energy must be positive at any r, the current is limited:

$$I < \frac{U\beta c}{2\ln\frac{b}{a}+1}\,. \tag{62}$$

To obtain higher currents, sheet beams are used. Then for a uniformly charged rectangular beam $|y| < a$, $|x| < w/2$, $w >> a$,

$$\varphi(y) = \frac{2\pi i}{v}\left(b - \frac{a}{2} - \frac{y^2}{2a}\right)\,, \tag{63}$$

where $2b$ is the vertical aperture of the conducting beam pipe (very wide in the x direction), and $i = I/w$ is the linear current density. Then instead of Eq. (61) and Eq. (62) we get

$$\Delta\gamma = \frac{\pi i a}{\beta I_0} \tag{64}$$

and

$$I < U\beta c\frac{w}{2b-a}\,. \tag{65}$$

Maximum Equilibrium Current Density

The equilibrium solution of Eq. (33) for constant focusing, $K = const$,

$$Ka - \frac{2I}{(\beta\gamma)^3 I_0}\frac{1}{a} = 0\,, \tag{66}$$

leads to a current density

$$j = \frac{I}{\pi a^2} = \frac{KI_0(\beta\gamma)^3}{2\pi}\,. \tag{67}$$

But there is no way to organize equal constant focusing in both transverse directions.[3] Therefore, a longitudinal magnetic field is used to keep the beam radius constant. As a uniform longitudinal magnetic field $B_z = B$ does not provide true focusing, we need to derive the equilibrium condition from the beginning.

At the beam rest frame the equilibrium condition is

$$eE_r + \frac{e}{c}v_\alpha B_z = -\frac{mv_\alpha^2}{r} \tag{68}$$

(the cylindrical coordinates r, α, z are used, and $v_\alpha << c$ is assumed). We will neglect the beam-induced contribution to the magnetic field B_z. According to Eq. (7) the radial electric field is linear in r. Therefore, choosing the velocities as $v_\alpha = -\omega r$, we can satisfy the equilibrium equation Eq. (68) for all r[4]:

---

[3] We can use a bending magnet with field index $n = 0.5$, but then the equilibrium trajectory is curved.

[4] In this case the beam is rotated as a rigid body.

$$\omega^2 - \omega_c\omega + \frac{\omega_p^2}{2} = 0\,, \tag{69}$$

where $\omega_c = \dfrac{eB}{mc}$ is the cyclotron frequency. The roots of Eq. (69) are

$$\omega = \frac{\omega_c}{2} \pm \sqrt{\frac{\omega_c^2}{4} - \frac{\omega_p^2}{2}}\,. \tag{70}$$

The maximum charge density takes place for the zero discriminant of Eq. (70), i. e. at $\omega_c = \omega_p\sqrt{2}$,

$$j_{\max} = \gamma\rho_0\beta c = \beta\gamma\frac{e}{mc}\frac{B_0^2}{8\pi}\,, \tag{71}$$

and is referred to as the Brillouin current density. For example, if $B_0 = 10$ kG and $\beta\gamma = 1$, then $j_{max} \approx 23$ kA/cm^2.

ELECTRON GUNS

The conditions of electron guns differ from those discussed above, because near the cathode particle velocities are low, and we can not use the paraxial approximation successfuly.

For an infinite-plane cathode at $x = 0$, the set of equations for a beam propagating in the positive x direction,

$$\begin{aligned} &\frac{d^2\varphi_1}{dx^2} = -4\pi\frac{j}{v}, \\ &\frac{mv^2}{2} - e\varphi_1 = 0, \end{aligned} \tag{72}$$

with initial conditions $\varphi_1(0) = 0$ and $d\varphi_1/dx(0) = 0$, has the solution

$$\varphi_1(x) = \left(-9\pi j\sqrt{\frac{m}{2e}}\right)^{2/3} x^{4/3} \tag{73}$$

(the current density j is negative). If the voltage at the gap d is $U = \varphi_1(d)$, the well-known Child's law

$$-j = \frac{1}{9\pi d^2}\sqrt{\frac{2e}{m}}U^{3/2}\,, \tag{74}$$

and the planar diode perveance

$$P = \frac{-jS}{U^{3/2}} = \frac{S}{9\pi d^2}\sqrt{\frac{2e}{m}}\,, \tag{75}$$

where S is the cathode area, can be easily obtained. In the general case we can not apply Eq. (74) for the finite transverse beam size, but now we'll find a special gun configuration, called Pierce's gun, for which Eqs. (74) and (75) are valid. Consider the two-dimensional problem with a beam in quadrant $x > 0$, $y < 0$, propagating as before in the positive x direction (see Fig. 1).

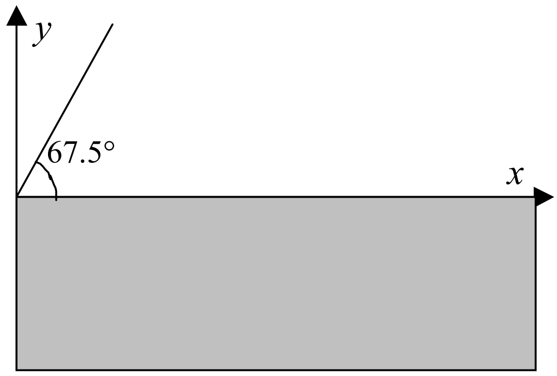

FIGURE 1. Geometry of the simplest Pierce's gun. The beam (gray) occupies the lower quadrant. The focusing zero-potential electrode (zero equipotential) is shown in the upper quadrant.

Let the potential in this quadrant be given by Eq. (73), so that the beam particles are moving as in the infinite diode case. The solution of the Laplace equation in the upper quadrant $x > 0$, $y > 0$, for the boundary conditions $\varphi(0,y) = 0$, $\varphi(x,0) = \varphi_1(x)$, can be obtained by the analytic continuation $\varphi(x,y) = \mathrm{Re}\varphi_1(x+iy)$. In particular, the zero equipotential equation

$$\mathrm{Re}(x+iy)^{4/3} = r^{4/3}\cos\left(\frac{4}{3}\alpha\right) = 0 \ , \tag{76}$$

where $x + iy = re^{i\alpha}$, gives $\alpha = 3\pi/8 = 67.5°$. The zero-potential (connected with the cathode at $x = y = 0$) electrode installed at this position provides the necessary potential inside the beam. The same electrode can limit the beam from the bottom also, and thus create a sheet beam. To construct the gun, we solved the inverse problem, finding the proper boundary conditions from the given potential. Solving the same problem in cylindrical coordinates, we can construct a parallel cylindrical beam and find the corresponding shape of the focusing electrodes. Using the solution for the spherical diode, we can construct a converging (conical) round beam.

For an arbitrary electrode configuration, the electron beam parameters can be calculated numerically.

In conclusion let us write down the set of equations for a laminar electron beam:

$$\begin{aligned} &m\mathbf{v} = \nabla S\,, \\ &\frac{m\mathbf{v}^2}{2} + e\varphi = 0\,, \\ &\nabla(\rho\mathbf{v}) = 0\,, \\ &\Delta\varphi = -4\pi\rho\,. \end{aligned} \tag{77}$$

Expressing all variables through the action S, we can obtain Spangenberg's equation,

$$\nabla\left\{\nabla S\,\Delta\left[(\nabla S)^2\right]\right\} = 0\,. \tag{78}$$

PROBLEMS

1. A hollow beam has inner radius a_1 and outer radius a_2. Find the space-charge-induced kinetic energy “spread” and the maximum beam current for gun voltage U. Consider also the limiting cases $a_1 \to 0$ and $a_1 \to a_2$.
2. A cylindrical beam collimator has opening radius b and length L. For the KV distribution find the maximum emittance of a round uncharged beam that can pass through the collimator.
3. Derive the equation for the variation of the vertical size of a laminar sheet beam (i. e. an infinitely wide beam). Compare the solution with the round beam case.
4. Find the maximum perveance of a round laminar beam for the collimator of Problem 2.
5. A coasting beam passed through a straight drift section of length L. Estimate the emittance degradation due to the nonlinearity of the transverse space-charge field. Compare this with the similar result for a bunched beam with a linear transverse space-charge field.
6. Consider the equilibrium condition and the maximum current density for a sheet (infinitely wide) beam in a longitudinal magnetic field.
7. A Brillouin beam leaves the solenoid passing through its short end-field region. Show that the beam rotation will be stopped outside the solenoid, i. e. the particles’ angular momenta with respect to the beam axis will vanish.
8. For a low current density, a beam has two equilibrium angular velocities in the longitudinal magnetic field. What is their meaning?

ACKNOWLEDGMENTS

The author thanks E. A. Perevedentsev for numerous stimulating discussions and advice, and also for help in preparation of these lectures.

REFERENCES

1. Lawson, J. D., The Physics of Charged-Particle Beams, Clarendon, Oxford, 1977.